Lagrange multipliers (continued)

Lagrange critical points condition is almost the same if there is one equality (regardless of # of variables)

Last time (2 variables, 1 equality)

Setup: Global max/min of $f(x,y)$, on the domain $\{g(x,y)=k\}$

Lagrange critical points:

(x,y) is a Lagrange critical point if the two vectors $\nabla f(x,y)$ and $\nabla g(x,y)$ are parallel.

This happens if EITHER (A) $\nabla g(x,y) = \langle 0,0 \rangle$
OR (B) There is a scalar λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.

Sample new setup (3 variables, 1 equality)

Setup: Global max/min of $f(x,y,z)$, on the domain $\{g(x,y,z)=k\}$

Lagrange critical points:

(x,y,z) is a Lagrange critical point if the two vectors $\nabla f(x,y,z)$ and $\nabla g(x,y,z)$ are parallel.

This happens if EITHER (A) $\nabla g(x,y,z) = \langle 0,0,0 \rangle$
OR (B) There is a scalar λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.

Ex Find the global max/min values of
$f(x,y,z) = x^4 - y^2 + z^2$ on the domain $x^2+y^2+z^2 \le 1$.

Sol. The domain has 1 inequality. So we look for
① Critical points of the original domain ($x^2+y^2+z^2 \le 1$)
② Lagrange critical points of the boundary ($x^2+y^2+z^2 = 1$)

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① Critical points of the original domain ($x^2+y^2+z^2 \le 1$)

This happens when $\nabla f(x,y,z) = \langle 0,0,0 \rangle$.
Since $\nabla f(x,y,z) = \langle 4x^3, -2y, 2z \rangle$, this is $= \langle 0,0,0 \rangle$ means
$x=y=z=0$. $(0,0,0)$ also satisfies $x^2+y^2+z^2 \le 1$
⇨ $\boxed{(0,0,0) \text{ is the critical point of the original domain.}}$

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② Lagrange critical points of the boundary ($x^2+y^2+z^2=1$)
$g(x,y,z) = x^2+y^2+z^2$.

This happens when EITHER Ⓐ $\nabla g(x,y,z) = \langle 0,0,0 \rangle$
OR Ⓑ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some scalar λ.

Ⓐ $\nabla g(x,y,z) = \langle 0,0,0 \rangle$
$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$, so this happens when $x=y=z=0$
But this doesn't satisfy $x^2+y^2+z^2=1$
⇨ $\boxed{\text{No Lagrange critical point in Case Ⓐ.}}$

(B) $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for some scalar λ.

This is $\langle 4x^3, -2y, 2z\rangle = \lambda\langle 2x, 2y, 2z\rangle = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$

⇨ Need to solve system of equation

Eq 1 … $4x^3 = 2\lambda x$
Eq 2 … $-2y = 2\lambda y$
Eq 3 … $2z = 2\lambda z$
Eq 4 … $x^2+y^2+z^2=1$.

Eq 2 says $-2y = 2\lambda y$. Both sides have $2y$ in them

Want to divide by $2y$ → ☺ can divide → $\boxed{-1=\lambda}$
→ ☹ can't divide because $\boxed{y=0}$

So Eq 2 ⇨ EITHER $\boxed{\lambda=-1}$ OR $\boxed{y=0}$

IF $\boxed{\lambda=-1}$ …
the system of eq. is

Eq 1 … $4x^3=-2x$
Eq 2 … $-2y=-2y$ ← satisfied ✓
Eq 3 … $2z=-2z$
Eq 4 … $x^2+y^2+z^2=1$

* Eq 1 ⇨ $4x^3+2x=0$ ⇨ $(4x^2+2)x=0$ ⇨ EITHER
$4x^2+2=0$ OR $x=0$
But $4x^2+2$ is never 0 (always ≥ 2) ⇨ $\boxed{x=0.}$

* Eq 3 ⇨ $\boxed{z=0.}$

* Eq 4 ⇨ $0^2+y^2+0^2=1$ ⇨ $\boxed{y=1}$ or $\boxed{y=-1}$

⇨ Lagrange critical points in this case: $\boxed{(0,1,0)}$, $\boxed{(0,-1,0)}$

IF $\boxed{y=0}$ ...

the system of eq is

Eq 1 ... $4x^3 = 2\lambda x$

Eq 2 ... $0 = 0$ ← satisfied ✓

Eq 3 ... $2z = 2\lambda z$

Eq 4 ... $x^2 + z^2 = 1$

Eq 3 says $2z = 2\lambda z$. Both sides have $2z$ in them

Want to divide by $2z$ → :) can divide → $\boxed{\lambda = 1}$

→ :(can't divide because $\boxed{z=0}$

So Eq 3 ⇒ EITHER $\boxed{\lambda=1}$ OR $\boxed{z=0}$

IF $\boxed{\lambda=1}$ ...

the system of eq is

Eq 1 ... $4x^3 = 2x$

Eq 2 ... $0 = 0$ ← satisfied ✓

Eq 3 ... $2z = 2z$ ← satisfied ✓

Eq 4 ... $x^2 + z^2 = 1$

* Eq 1 ⇒ $4x^3 - 2x = 0$ ⇒ $2x(2x^2 - 1) = 0$ ⇒ EITHER $x=0$ OR $x^2 = \frac{1}{2}$

⇒ EITHER $\boxed{x=0}$ OR $\boxed{x = \frac{1}{\sqrt{2}}}$ OR $\boxed{x = -\frac{1}{\sqrt{2}}}$

$\boxed{x=0}$ → Eq4 → $z^2 = 1$ → EITHER $\boxed{z=1}$ OR $\boxed{z=-1}$

$\boxed{x=\frac{1}{\sqrt{2}}}$ → Eq 4 → $z^2 = \frac{1}{2}$ → EITHER $\boxed{z = \frac{1}{\sqrt{2}}}$ OR $\boxed{z = -\frac{1}{\sqrt{2}}}$

$\boxed{x=-\frac{1}{\sqrt{2}}}$ → Eq 4 → $z^2 = \frac{1}{2}$ → EITHER $\boxed{z = \frac{1}{\sqrt{2}}}$ OR $\boxed{z = -\frac{1}{\sqrt{2}}}$

⇒ Lagrange critical points in this case are $\boxed{(0,0,1)}$, $\boxed{(0,0,-1)}$, $\boxed{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)}$

IF $\boxed{z=0}$ ...

the system of eq is

Eq 1 ... $4x^3 = 2\lambda x$

Eq 2 ... $0=0$ ⇜ satisfied ✓

Eq 3 ... $0=0$ ⇜ satisfied ✓

Eq 4 ... $x^2=1$

$*$ Eq 4 ⇨ EITHER $\boxed{x=1}$ OR $\boxed{x=-1}$

⇨ Lagrange critical points in this case are $\boxed{(1,0,0)}$, $\boxed{(-1,0,0)}$

Summary: 10 Lagrange critical points, $\boxed{(0,1,0)}$, $\boxed{(0,-1,0)}$, $\boxed{(0,0,1)}$, $\boxed{(0,0,-1)}$, $\boxed{\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)}$, $\boxed{\left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)}$, $\boxed{(1,0,0)}$, $\boxed{(-1,0,0)}$

Types			(x,y,z)	$f(x,y,z)$
Critical			$(0,0,0)$	$f(0,0,0) = 0^4-0^2+0^2 = \boxed{0}$
Boundary	Lagrange critical	A	N/A	
		B	$(0,1,0)$	$f(0,1,0) = 0^4-1^2+0^2 = \boxed{-1}$
			$(0,-1,0)$	$f(0,-1,0) = 0^4-(-1)^2+0^2 = \boxed{-1}$
			$(0,0,1)$	$f(0,0,1) = 0^4-0^2+1^2 = \boxed{1}$
			$(0,0,-1)$	$f(0,0,-1) = 0^4-0^2+(-1)^2 = \boxed{1}$
			$\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$	$f\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^4-0^2+\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{3}{4}}$
			$\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$	$f\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^4-0^2+\left(-\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{3}{4}}$
			$\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$	$f\left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)^4-0^2+\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{3}{4}}$
			$\left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$	$f\left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)^4-0^2+\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{3}{4}}$
			$(1,0,0)$	$f(1,0,0) = 1^4-0^2+0^2 = \boxed{1}$
			$(-1,0,0)$	$f(-1,0,0) = (-1)^4-0^2+0^2 = \boxed{1}$

⇨ Global max value : $\boxed{1}$ (at $(1,0,0), (-1,0,0), (0,0,1), (0,0,-1)$)
Global min value : $\boxed{-1}$ (at $(0,1,0), (0,-1,0)$)

The definition of Lagrange critical points change a bit if the domain has 2 equalities.

Setup (3 variables, 2 equalities)

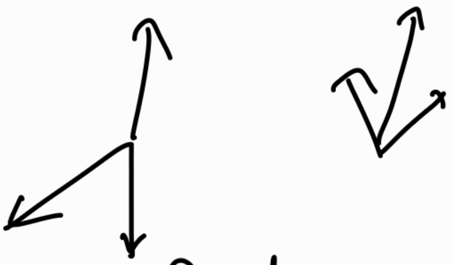
Setup: Global max/min of $f(x,y,z)$, on the domain $\{g(x,y,z)=k$ and $h(x,y,z)=\ell\}$.

Lagrange critical points :

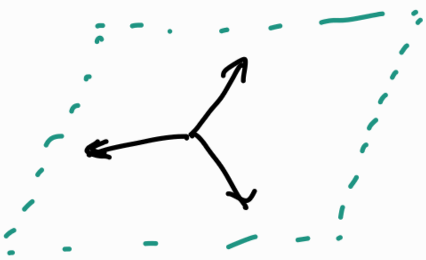
(x,y,z) is a Lagrange critical point if the three vectors $\nabla f(x,y,z)$, $\nabla g(x,y,z)$, $\nabla h(x,y,z)$ lie in some plane. (coplanar)

This happens if EITHER (A1) $\nabla g(x,y,z) = \langle 0,0,0\rangle$
OR (A2) $\nabla h(x,y,z) = \langle 0,0,0\rangle$
OR (B) There are scalars λ, μ such that
$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z).$$

Think about the condition of three 3D vectors lying in a single plane. This is quite a restriction: if you choose three random 3D vectors, most of the time they are not contained in some plane. We say 3 vectors are coplanar if they lie on some plane.



Random



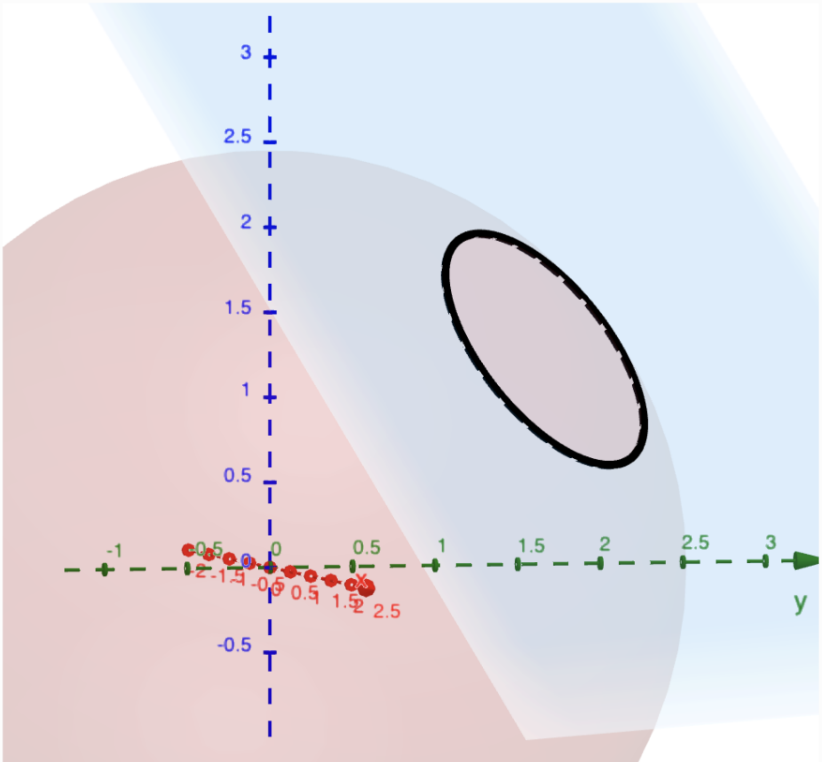
Coplanar

In the 1 equality case of Lagrange critical points, the condition of $\nabla f(x,y,z)$ & $\nabla g(x,y,z)$ parallel can be thought of as both vectors lying on some line. In this way 1 equality case & 2 equalities case are analogous.

Ex Find the global max/min values of

$f(x,y,z) = x^2+2y+2z$ on the domain

$\{x^2+y^2+z^2=6,\ x+y+z=4\}$.



The solid circle is the domain.

Sol. The domain is $\{g(x,y,z)=6,\ h(x,y,z)=4\}$ where

$g(x,y,z) = x^2+y^2+z^2$, $h(x,y,z) = x+y+z$

The Lagrange critical points are when

EITHER (A1) $\nabla g(x,y,z) = \langle 0,0,0\rangle$

OR (A2) $\nabla h(x,y,z) = \langle 0,0,0\rangle$

OR (B) $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$ for some λ, μ.

$\textcircled{A_1}$ $\nabla g(x,y,z) = \langle 0,0,0 \rangle$

$\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$, so it is $= \langle 0,0,0 \rangle$ exactly if $x=y=z=0$.

But this does not satisfy $x^2+y^2+z^2=6$ or $x+y+z=4$.

$\Rightarrow$ $\boxed{\text{No Lagrange critical points in Case } A_1}$

$\textcircled{A_2}$ $\nabla h(x,y,z) = \langle 0,0,0 \rangle$

$\nabla h(x,y,z) = \langle 1,1,1 \rangle$, so it is never equal to $\langle 0,0,0 \rangle$.

$\Rightarrow$ $\boxed{\text{No Lagrange critical points in Case } A_2}$

$\textcircled{B}$ $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$ for some λ, μ.

$\nabla f(x,y,z) = \langle 2x, 2, 2 \rangle$, so this equation + domain equations give a system of equations

Eq1 ... $2x = 2\lambda x + \mu$

Eq2 ... $2 = 2\lambda y + \mu$

Eq3 ... $2 = 2\lambda z + \mu$

Eq4 ... $x^2+y^2+z^2=6$

Eq5 ... $x+y+z=4$

Eq2 − Eq3 $\Rightarrow$ $0 = 2\lambda y - 2\lambda z = 2\lambda(y-z)$.

This means EITHER $\boxed{\lambda=0}$ OR $\boxed{y-z=0}$

IF $\boxed{\lambda=0}$ ... the system of eq becomes

Eq1 ... $2x = \mu$

Eq2 ... $2 = \mu$ $\rightsquigarrow$ $\boxed{\mu=2}$ and $2x=2$, or $\boxed{x=1}$

Eq3 ... $2 = \mu$

Eq4 ... $x^2+y^2+z^2=6$

Eq5 ... $x+y+z=4$

Plugging into Eq4 and Eq5 $\Rightarrow$ Eq 4 ... $y^2+z^2=5$

Eq 5 ... $y+z=3$

Eq5 $\Rightarrow$ $y=3-z$ $\xrightarrow{\text{Eq 4}}$ $(3-z)^2+z^2=5$, or

$2z^2-6z+9=5$, or

$2z^2-6z+4=0$, or

$z^2-3z+2=0$, or

$(z-1)(z-2)=0$.

$\Rightarrow$ EITHER $\boxed{z=1}$ OR $\boxed{z=2}$

$\boxed{y=2}$ $\boxed{y=1}$

$\Rightarrow$ Lagrange critical points are $\boxed{(1,1,2)}$, $\boxed{(1,2,1)}$

IF $\boxed{y-z=0}$ ... Plugging into Eq4 and Eq5

$\Rightarrow$ Eq 4 ... $x^2+2y^2=6$

Eq5 ... $x+2y=4$

Eq5 $\Rightarrow$ $x=4-2y$ $\xrightarrow{\text{Eq 4}}$

$(4-2y)^2+2y^2=6$, or

$6y^2-16y+16=6$, or

$6y^2-16y+10=0$, or

$3y^2-8y+5=0$, or

$(3y-5)(y-1)=0$.

$\Rightarrow$ EITHER $\boxed{y=\frac{5}{3}}$ OR $\boxed{y=1}$

$\boxed{z=\frac{5}{3}}$ $\boxed{z=1}$

$\boxed{x=4-\frac{10}{3}=\frac{2}{3}}$ $\boxed{x=4-2=2}$

$\Rightarrow$ Lagrange critical points are $\boxed{\left(\frac{2}{3},\frac{5}{3},\frac{5}{3}\right)}$, $\boxed{(2,1,1)}$

Lagrange critical points in Case B : $\boxed{(1,1,2)}$, $\boxed{(1,2,1)}$

$\boxed{\left(\frac{2}{3},\frac{5}{3},\frac{5}{3}\right)}$, $\boxed{(2,1,1)}$

As there is no boundary (no inequalities in the domain),

the table is..

Types		(x,y,z)	$f(x,y,z)$
Lagrange critical	A1	N/A	
	A2	N/A	
	B	$(1,1,2)$	$f(1,1,2)=1^2+2\times1+2\times2=\boxed{7}$
		$(1,2,1)$	$f(1,2,1)=1^2+2\times2+2\times1=\boxed{7}$
		$\left(\frac{2}{3},\frac{5}{3},\frac{5}{3}\right)$	$f\left(\frac{2}{3},\frac{5}{3},\frac{5}{3}\right)=\left(\frac{2}{3}\right)^2+2\times\frac{5}{3}+2\times\frac{5}{3}=\boxed{\frac{64}{9}}$
		$(2,1,1)$	$f(2,1,1)=2^2+2\times1+2\times1=\boxed{8}$
Boundary		N/A	

$\Rightarrow$ Global max value : $\boxed{8}$

Global min value : $\boxed{7}$

This gives a way to deal with global max/min on domains like 3 variables, 2 inequalities but the process is annoying. (We already learned everything to do it, it's just that the process is long.) That's going to come next time.

For reference: Lagrange critical points for n variables m equalities.

Setup: Find global max/min value of

$$f(x_1, \dots, x_n) \text{ on the domain } \left\{ \begin{array}{l} g_1(x_1, \dots, x_n) = k_1, \\ g_2(x_1, \dots, x_n) = k_2, \\ \vdots \\ g_m(x_1, \dots, x_n) = k_m \end{array} \right\}$$

$\Rightarrow$ $(x_1, \dots, x_n)$ is a Lagrange critical point if

EITHER (A1) $\nabla g_1(x_1, \dots, x_n) = \langle 0, \dots, 0 \rangle$

OR (A2) $\nabla g_2(x_1, \dots, x_n) = \langle 0, \dots, 0 \rangle$

... ...

OR (A_m) $\nabla g_m(x_1, \dots, x_n) = \langle 0, \dots, 0 \rangle$

OR (B) There exist scalars $\lambda_1, \dots, \lambda_m$ such that

$$\nabla f(x_1, \dots, x_n) = \lambda_1 \nabla g_1(x_1, \dots, x_n) + \cdots + \lambda_m \nabla g_m(x_1, \dots, x_n)$$